Some Aspects of Multigrid for Mixed Discretizations*

J. E. Dendy, Jr.[1] and J. D. Moulton[1]

Mathematical Modeling and Analysis, Theoretical Division, Los Alamos National Laboratory, Mail Stop B-284,Los Alamos, NM 87544, U.S.A.

Abstract. A broad class of discretizations of the diffusion operator is based on its first order form, allowing the rigorous enforcement of many desirable physical properties of the continuous model. In this research we investigate the development of multilevel solvers for the *local* or *hybrid* forms of these discretizations on logically rectangular quadrilateral meshes. In this case, the local elimination of flux leads to a system that contains both cell- and edge-based scalar unknowns. Based on this natural partitioning of the system we develop approximate reduced systems that reside on a single logically rectangular grid. Each such approximate reduced system, formed as an approximate Schur complement or as a variational product, are used as the first coarse-grid in a multigrid hierarchy or as a preconditioner for Krylov based methods.

1 Introduction

Mixed discretizations for the solution of the diffusion equation, which are based on the first-order form,

$$\nabla \cdot \mathbf{F} = Q(\mathbf{r}) \tag{1a}$$

$$\mathbf{F} = -\mathcal{D}(\mathbf{r}) \nabla \phi \,, \tag{1b}$$

are currently popular because they rigorously enforce important physical properties, such as mass balance and continuity of normal flux. Examples of such discretizations include mixed finite element methods (e.g., [5]) and support operator methods (e.g., [6]). However, the first order form defines a saddle point problem, and hence, its discretization leads to an indefinite linear system. In the hybrid or local version of these discretizations (e.g., [3,**?**]), it is possible to eliminate the normal flux, locally on each cell, to obtain a sparse system in the scalar unknowns. Unfortunately, this reduced system has both cell and edge unknowns; hence, the direct application of existing robust multigrid algorithms for logically rectangular grids, such as "black box multigrid" [1], is problematic.

In this work we consider logically rectangular meshes composed of quadrilaterals. Thus a natural approach to solving these systems is to approximately

* Report No. LA-UR 99-4307. This research is supported by the U.S. Department of Energy under contract W-7405-ENG-36.

eliminate either edge- or cell-based unknowns to obtain a reduced system on the corresponding logically rectangular grid. This reduced system may act as a preconditioner for Krylov based methods, inverted approximately by a single V-cycle of a robust multigrid algorithm, or it may be the first coarse-grid in a multigrid hierarchy.

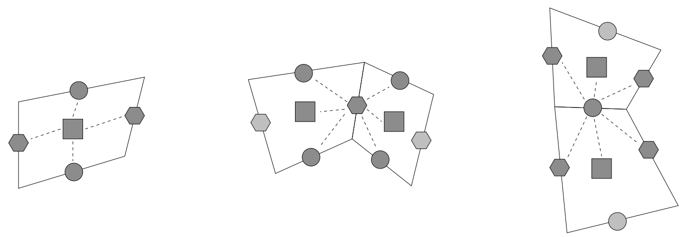
Specifically, we investigate four possible approximate reduced systems for the Morel diffusion scheme [3], a support operator method, on uniform parallelogram meshes. In this case this scheme is symmetric positive definite, and its characteristics are representative of this class of discretizations. In Sect. 2 we provide a brief discussion of the sparsity structure of this scheme and the definitions of the scalar unknowns. The reduced systems are developed in Sect. 3. We begin this development, in Sect. 3.1, with a review of the 5-point approximate Schur complement on the cell-based unknowns that appeared in [3]. Although, this method is adequate for modestly distorted meshes, it unfortunately performs poorly for highly skewed meshes.

An alternative to the Schur complement approach is considered in Sect. 3.2, namely the construction of the approximate reduced system through a variational product. This is motivated by both the mathematical consideration that variational coarsening provides the optimal reduced operator in the sense that it minimizes the error in the range of the interpolation [2], and the success of operator-induced variational coarsening in "black box multigrid". In the first case, we adopt the the interpolation that arises in the aforementioned approximate Schur complement to derive a 9-point cell-based operator. This method is approximately twice as fast as the previous method, and hence, it still has unacceptably slow convergence highly skewed meshes. This poor performance, inspired our interest in improved operator-induced interpolation. Specifically, we consider a larger cell to edge based interpolation (6-point), which leads to a 25-point cell-based reduced operator. Similarly, we consider the exact elimination of the cell-based unknowns followed by the variational product involving a 4-point interpolation, which leads to a 15-point operator. We demonstrate that these methods exhibit good convergence, even in the presence of severe skewing.

2 Mixed Discretizations

We consider a $(N_x \times N_y)$ logically rectangular grid of quadrilaterals. The hybrid or local discretization of (1) generates an indefinite linear system with a favorable sparsity structure. In particular, it is to eliminate the flux locally on each cell leading to a system for the cell- and edge-based scalar unknowns. Typically, the cell-based unknowns represent either values of the scalar ϕ at a point in the cell or its integral average over the cell. Similarly, the edge-based unknowns represent either point values of the scalar ϕ on the edge or its integral average along the edge. Thus we define the cell-based vector, $\boldsymbol{\phi}_h^T = [\ldots, \phi_{i,j}, \ldots]$, $i = 1, \ldots, Nx$, $j = 1, \ldots, Ny$, and the edge-based vector

$\boldsymbol{\mu}_h^T = [\boldsymbol{u}_h^T, \boldsymbol{v}_h^T]$ by $\boldsymbol{u}_h = [\ldots, \phi_{i+\frac{1}{2},j}, \ldots]$, $i = 1, \ldots, (Nx+1)$, $j = 1, \ldots, Ny$ and $\boldsymbol{v}_h = [\ldots, \phi_{i,j+\frac{1}{2}}, \ldots]$ $i = 1, \ldots, Nx$, $j = 1, \ldots, (Ny+1)$. Schematically the sparsity structure for the Morel diffusion scheme [3] is shown in Fig. 1.



(a) (b) (c)

Fig. 1. Schematic of (a) cell-based, (b) vertical and (c) horizontal edge-based stencils. Dashed lines indicate nonzero connections for a logically rectangular grid.

3 Approximate Reduced Systems

The presence of both cell and edge unknowns generates a natural block partitioning of the discrete linear system,

$$\mathcal{S}_{(\phi,\mu)} \begin{bmatrix} \boldsymbol{\mu}_h \\ \boldsymbol{\phi}_h \end{bmatrix} = \begin{bmatrix} A_{\mu\mu} & A_{\mu\phi} \\ A_{\mu\phi}^T & A_{\phi\phi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_h \\ \boldsymbol{\phi}_h \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}_\mu \\ \boldsymbol{Q}_\phi \end{bmatrix}, \tag{2}$$

where the edge blocks may be written

$$A_{\mu\mu} = \begin{bmatrix} A_{uu} & A_{uv} \\ A_{uv}^T & A_{uv} \end{bmatrix} \qquad A_{\mu\phi} = [A_{u\phi} | A_{v\phi}] \,. \tag{3}$$

Noting that each type of unknown in $[\boldsymbol{\phi}_h, \boldsymbol{u}_h, \boldsymbol{v}_h]$ resides on a logically rectangular grid, and that the system is sparse motivates the investigation of reduced systems that may be treated by existing robust multilevel methods.

3.1 A Cell-Based Approximate Schur Complement

The multigrid scheme in [3] is derived as follows: First, an approximate flux continuity condition for the cell-edges is formed by collapsing the edge-based stencils to form $[\widetilde{A_{\mu\mu}} | \widetilde{A_{\mu\phi}}]$ where $\widetilde{A_{\mu\mu}}$ is a diagonal approximation and $\widetilde{A_{\mu\phi}}$ has consistently modified weights. (see Fig. 2). Therefore, forming the Schur complement generates a 5-point cell-based reduced operator. This naturally leads to a multilevel method: employ one relaxation sweep on the full system (2), solve for corrections to $\boldsymbol{\phi}_h$ with one V-cycle of black box multigrid [1] on the 5-point cell-based reduced operator, and interpolate these corrections.

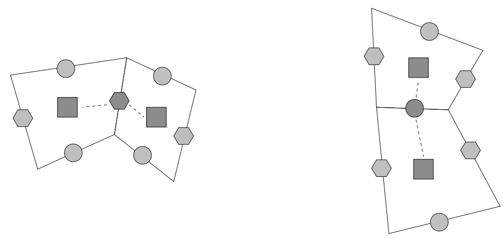


Fig. 2. Two-point interpolation (a) vertical (b) horizontal edge-based unknowns.

3.2 Variational Coarsening

The popularity of variational coarsening within the multigrid community arises primarily from its mathematical foundation. Specifically, the variationally formed reduced operator is optimal in the sense that it minimizes the error in the range of the interpolation [2]. We form the operator as the product

$$L_\alpha = (I_\alpha^\beta)^T L_{(\alpha,\beta)} I_\alpha^\beta \,. \tag{4}$$

where $L_{(\alpha,\beta)}$ is a block system partitioned into the unknowns of type α and β, I_α^β is an interpolation operator for the β unknowns in terms of the α unknowns and L_α is the variational operator on the α unknowns. If the interpolation I_α^β is exact, then this variational product generates the Schur complement. However, if an approximate interpolation is used the variational product differs significantly from the Schur complement.

A Cell-Based 9-point Operator The approximation to $[A_{\mu\mu}|A_{\mu\phi}]$ that is described in the previous section (Fig. 2) defines an approximate interpolation operator

$$\widetilde{I}_\phi^\mu = \begin{bmatrix} -\left(\widetilde{A_{\mu\mu}}\right)^{-1}\widetilde{A_{\mu\phi}} \\ I_\phi \end{bmatrix} . \tag{5}$$

A novel implementation of this interpolation introduces fictitious identity equations for cell-vertex unknowns $\phi_{i-\frac{1}{2},j-\frac{1}{2}}$. If the coarse grid unknowns begin at $\phi_{1,1}$, then the application of "black box" is automatic: the interpolation is given in (5) and the coarse grid operator on the cell-based unknowns is the operator-induced variational coarse-grid operator. The relaxation on the finest gird is the same as that employed in [3]; the relaxation on the coarser grids is alternating red-black line relaxation.

A Cell-Based 25-point Operator The multigrid method based on the 9-point cell-based reduced operator also suffers a deterioration in the convergence factor for highly skewed grids. For parallelogram grids and constant coefficients, the interpolation (5) is second order accurate and the coarse grid operator is optimal. It is natural, therefore, to consider an interpolation with a bigger stencil. To this end we construct an approximation to the product $A_{\mu\mu}^{-1}A_{\mu\phi}$ rather than approximating $A_{\mu\mu}$ and $A_{\mu\phi}$ separately. Thus, the interpolation operator,

$$\widehat{I}_{\phi}^{\mu} = \begin{bmatrix} -\widehat{\left(A_{\mu\mu}\right)^{-1}A_{\mu\phi}} \\ I_{\phi} \end{bmatrix} = \begin{bmatrix} -E_{\mu\phi} \\ I_{\phi} \end{bmatrix}, \tag{6}$$

is obtained by defining an approximate system of local equations and extracting the necessary column from its inverse. The 7×7 linear system is created by considering the exact equation at a particular edge unknown followed by local approximations to the neighboring equations. The sparsity structure of the resulting interpolation operator is shown in Fig. 3.

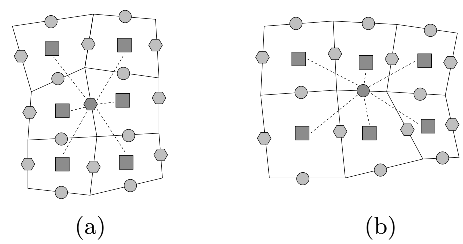


Fig. 3. Schematic of (a) cell-based, (b) vertical and (c) horizontal edge-based stencils. Dashed lines indicate nonzero connections for a logically rectangular grid.

An Edge-Based 15-point Operator Since the $A_{\phi\phi}$ is diagonal we may eliminate $\phi_{i,j}$ exactly to obtain a sparse system that involves only the edge unknowns. There are then two sets of equations centered at cell edges,

$$\mathcal{S}_{\mu}\boldsymbol{\mu}_h = \begin{bmatrix} B_{uu} & B_{uv} \\ A_{uv}^T & B_{vv} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{Q}}_u \\ \widetilde{\boldsymbol{Q}}_v \end{bmatrix}. \tag{7}$$

We consider the approximate interpolation,

$$\widetilde{I}_{v}^{u} = \begin{bmatrix} -\left(\widetilde{B_{uu}}\right)^{-1}B_{uv} \\ I_{\phi} \end{bmatrix} \tag{8}$$

where $\widetilde{B_{uu}}$ is a diagonal approximation of B_{uu} and which is shown in Fig. 4. The variational product, Equation (4), generates an approximate 15-point operator on the horizontal edges.

4 Linear Solvers

Multilevel methods for reduced systems that are derived as either approximate Schur complements or variational products are readily developed because all of the necessary components are present. In contrast defining a

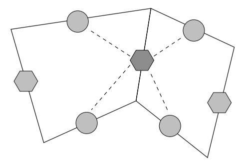


Fig. 4. Schematic of the edge to edge interpolation.

preconditioner for conjugate gradients follows naturally in the case of the approximate Schur complement but some care is required in the case of the variational product. For example, consider the approximations to (2) that lead to a 5-point cell-based approximate Schur complement and and let r_ϕ and r_μ denote the respective residuals. The preconditioner is

$$\widetilde{\mathcal{S}_{(\phi,\mu)}} \begin{bmatrix} \triangle \boldsymbol{\mu}_h \\ \triangle \boldsymbol{\phi}_h \end{bmatrix} = \begin{bmatrix} \widetilde{A_{\mu,\mu}} & \widetilde{A_{\mu,\phi}} \\ \widetilde{A_{\mu,\phi}}^T & A_{\phi,\phi} \end{bmatrix} \begin{bmatrix} \triangle \mu \\ \triangle \phi \end{bmatrix} = \begin{bmatrix} r_\mu \\ r_\phi \end{bmatrix}, \tag{9}$$

with the updates $\mu \to \mu + \triangle\mu$ and $\phi \to \phi + \triangle\phi$. Thus,

$$\widetilde{\mathcal{S}_\phi} \triangle\phi = \Big(A_{\phi,\phi} - \widetilde{A_{\mu,\phi}}^T \big(\widetilde{A_{\mu,\mu}}\big)^{-1} \widetilde{A_{\mu,\phi}} \Big) \triangle\phi = r_\phi - \widetilde{A_{\mu,\phi}}^T \big(\widetilde{A_{\mu,\mu}}\big)^{-1} r_\mu \tag{10}$$

and $\triangle\mu = \big(\widetilde{A_{\mu,\mu}}\big)^{-1} \big[r_\mu - \widetilde{A_{\mu,\phi}} \triangle\phi \big]$. In contrast, for the 9-point variational case the approximations are introduced in the interpolation (5) and we want to generate the corrections $\triangle\phi$ with $\widetilde{\mathcal{S}}_\phi^{(v)}$ given by (4). It is not immediately obvious that this the variational operator can be used in this way, however, one can show that $\widetilde{\mathcal{S}}_\phi^{(v)}$ is the Schur complement of

$$\mathcal{S}_{(\phi,\mu)}^{(*)} = \begin{bmatrix} A_{\mu,\mu} & A_{\mu,\phi} \\ A_{\mu,\phi}^T & A_{\phi,\phi}^{(*)} \end{bmatrix} \tag{11}$$

where

$$A_{\phi,\phi}^{(*)} = A_{\phi\phi} + \big(A_{\mu\phi}^T A_{\mu\mu}^{-1} - \widetilde{A_{\mu\phi}}^T \big(\widetilde{A_{\mu\mu}}\big)^{-1} \big) A_{\mu\mu} \big(A_{\mu\mu}^{-1} A_{\mu\phi} - \big(\widetilde{A_{\mu\mu}}\big)^{-1} \widetilde{A_{\mu\phi}} \big)$$

which is symmetric, and moreover, $\widetilde{\mathcal{S}}_{(\phi,\mu)}^{(*)} > 0$ if $\mathcal{S}_{(\phi,\mu)} > 0$ and $A_{\mu\mu} > 0$. Thus corrections based on $\widetilde{\mathcal{S}}_\phi^{(v)}$ are readily defined.

5 Numerical Results

All the numerical results reported here employ a constant diffusion coefficient, $D \equiv \frac{2}{11}$, on a 49×49 logically rectangular grid of parallelograms. The "vertical" edges of the parallelograms are perturbed from the vertical by an angle θ, yielding challenging problems as θ approaches $90°$. In Table 1 we report the average convergence factor for ten cycles, computed in terms of the discrete L^2 norm of the residual. Specifically, we display results for the four reduced operators that were developed in Sect. 3, with each method denoted by the number of points in its reduced system. Results for the "5-point" and "9-point" methods are presented for both V-cycle multigrid and preconditioned conjugate gradient. For the "15-point" and and "25-point" the results are for two-grid methods in which the reduced system is solved with diagonally scaled conjugate gradient.

The edge-based "15-point" system is the least sensitive to θ and provides the best overall convergence factors. However, the cost of inverting this reduced operator is prohibitive. Thus, we are investigating 9-point approximations to this variationally derived operator that are based on a flux analysis, as more naive lumping approaches have been found to be inadequate. We also note that for both the "5-point" and "9-point" reduced systems, the preconditioned conjugate gradient iterations exhibited better convergence factors with a significantly weaker dependence on θ as it approached 90^o than the corresponding multigrid methods.

Table 1. Convergence Factors, ρ, for various reduced systems

θ	5-pnt.	5-pnt.(PCG)	9-pnt.	9-pnt.(PCG)	25-pnt.	15-pnt.
0°	0.04	0.08	0.05	0.05	0.006	0.002
45°	0.64	0.66	0.33	0.40	0.06	0.02
60°	0.81	0.78	0.57	0.55	0.26	0.07
72°	0.93	0.85	0.80	0.74	0.60	0.27
80°	0.96	0.85	0.93	0.82	0.66	0.43

References

1. Dendy, Jr., J. E.: Black box multigrid. J. Comp. Phys. **48** (1982) 366–386
2. McCormick, S.: An algebraic theory for multigrid methods for variational problems, SIAM J. Numer. Anal. **25** (1988) 91–110
3. Morel, J. E., Dendy, Jr., J. E., Hall, M., White S.: A cell-centered Lagrangian-mesh diffusion differencing scheme. J. Comp. Phys. **103** (1992) 286–299
4. Morel, J. E., Roberts, R. M, Shashkov, M.: A local support-operators diffusion discretization scheme for quadrilateral r-z meshes. J. Comp. Phys. **144** (1998) 17–51
5. Raviart P. A., Thomas J. M.: A mixed finite element method for 2-nd order elliptic problems. in *Mathematical Aspects of the Finite Element Method*. Volume 606 of Lecture Notes in Mathematics. Galligani I., Magenes, editors. Springer Verlag, (1977) 292–315
6. Shashkov M., Steinberg S.: Solving diffusion equations with rough coefficients in rough grids. J. Comp. Phys. **129** (1996) 383–405